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The Application of a Local Similarity Concept in Solving the Radial Flow Problem

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A technique is described for solving the radial flow problem of underground water by the application of a similarity transformation when all the requirements for the application of such a transformation are not met. The transformed equation involves a stray time variable which is interpreted as a system parameter. When this parameter is properly interpreted, the solutions of the transformed system are very good numerical approximations for the solutions of the original system. The technique was checked for many different nonconstant diffusivities. The advantage of solving the transformed system is the great saving in computer time, since no marching process is involved.

1. INTRODUCTION

A useful mathematical technique in the study of radial flow of underground water or of the radial heat conduction in a circular plate is the application of Boltzmann's similarity transformation [1]. To apply this transformation rigorously to the radial flow problem several requirements must be satisfied. One of these requirements is that the dependent variable (such as the moisture content or the temperature of the plate) must have a logarithmic singularity when the radius is zero. There are important radial flow problems which do not satisfy this boundary condition but which do satisfy the other requirements for the application of the similarity transformation. These problems will be the subject of this paper.

A Boltzmann type transformation is applied to the partial differential equation and the corresponding initial and boundary conditions describing the radial flow. The resulting system is an ordinary differential equation whose solution must satisfy two boundary conditions. Since all of the requirements necessary for the rigorous application of the transformation were not met, the time variable will occur either in the differential equation or in one or both of the boundary

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conditions. In this paper the idea of local similarity is that the time variable T in the transformed system plays the role of a fixed parameter. This fixed parameter is related, in some manner, to an actual time T_1 in the original system. Thus, if a particular moisture or temperature distribution with respect to the radius is desired for a given time T_1, the transformed two-point boundary value problem is solved for the value of the parameter $T = f(T_1)$. The advantage of this method is a great saving in computer time since a marching process is not needed to solve the transformed system. Therefore, one of the important objectives of this paper is the determination of the relationship $T = f(T_1)$ which will give an acceptable approximation to the original system.

In Section 2 the physical problem is briefly formulated. Two possible Boltzmann type transformations are formulated in Section 3, and the corresponding transformed systems are given. In Section 4 the solutions for the linear problem (constant diffusivity) are listed for both the original and transformed systems. Several possible choices for the relationship $T = f(T_1)$ are given in Section 5. In Section 6 the numerical solutions for the two systems are described and compared for several different forms of the diffusivity. Finally, Section 7 lists the conclusions arrived at from this study.

2. Physical Problem

Since this study originated from the investigation of the radial flow of underground water [2], the physical problem will be expressed in terms of soil moisture variables. The differential equation describing the plane radial movement of soil moisture through an isotropic medium without gravity is given as

$$\theta_t = \frac{1}{r} [r\bar{D}(\theta)\, \theta_r]_r \tag{1}$$

where r is the radial distance, t the time, θ the volume of soil water per volume of soil, and $\bar{D} = \bar{D}(\theta)$, the soil moisture diffusivity.

The auxiliary conditions which θ must satisfy are

$$\begin{aligned} \theta(r, 0) &= \theta_0, \qquad r > a > 0, \\ \theta(\infty, t) &= \theta_0, \qquad t \geqslant 0, \\ \theta(a, t) &= \theta_1, \qquad t \geqslant 0, \end{aligned} \tag{2}$$

where a, θ_1, and θ_0 are positive constants. When $\theta_1 > \theta_0$, moisture moves away from the boundary at $r = a$ as t increases and is thus known as the "source problem." The "sink problem" is the name used when $\theta_1 < \theta_0$.

For convenience the soil moisture is normalized as follows:

$$\lambda = \lambda(r, t) = \frac{\theta - \theta_1}{\theta_0 - \theta_1}, \qquad \bar{D}(\theta) = D^*(\lambda). \tag{3}$$

Based on the physics of the problem it is reasonable to impose the following conditions on the diffusivity over the range of values $0 \leqslant \lambda \leqslant 1$:

$$\begin{aligned} &0 < D^*_{\min} \leqslant D^*(\lambda) \leqslant D^*_{\max}, \\ &dD^*/d\lambda \text{ continuous over } (0, 1), \\ &dD^*/d\lambda \leqslant 0, \qquad \text{if} \quad \theta_1 > \theta_0, \\ &dD^*/d\lambda \geqslant 0, \qquad \text{if} \quad \theta_1 < \theta_0, \end{aligned} \tag{4}$$

where $D^*_{\min}$ and $D^*_{\max}$ are constants.

The system defined by (1) and (2) is nondimensionalized by making use of the nondimensional terms

$$\begin{aligned} T &= \frac{D^*_{\min} t}{a^2}, \\ R &= \frac{r}{a}, \\ y(R, T) &= \lambda(r, t), \\ D(y) &= \frac{D^*(\lambda)}{D^*_{\min}}, \\ A &= \frac{D^*_{\max}}{D^*_{\min}} > 1, \end{aligned} \tag{5}$$

where T is a nondimensional time and R is a nondimensional radius. Combining (1), (2), (3), and (5) gives

$$y_T = \frac{1}{R} [RD(y)\, y_R]_R \tag{6}$$

where

$$\begin{aligned} y(R, 0) &= 1, \qquad R > 1, \\ y(\infty, T) &= 1, \qquad T \geqslant 0, \\ y(1, T) &= 0, \qquad T \geqslant 0. \end{aligned} \tag{7}$$

The problem to be analyzed in this paper is defined by (6) and (7), with diffusivities satisfying the conditions in (4).

3. Transformations

One Boltzmann type transformation which can be applied to (6) and (7) is given by

$$x = \frac{R^2}{4T}, \qquad y(x) = y(R, T). \tag{8}$$

Substituting (8) into (6) and (7) results in the following two-point boundary value problem:

$$\begin{aligned} [xD(y)\,y']' + xy' &= 0, \\ y(\infty) = 1, &\qquad T > 0, \\ y\left(\frac{1}{4T}\right) = 0, &\qquad T > 0, \end{aligned} \tag{9}$$

where the primes indicate differentiation with respect to x. Since T occurs explicitly in one of the boundary conditions in (9), the system described by (6) and (7) does not possess a similarity solution of the type specified in (8). But all is not lost; if T is "properly interpreted," the numerical solution of (9) can be shown to be an excellent approximation to the solution of (6) and (7) for a large family of different diffusivities. The comparison of these numerical solutions will be discussed in Section 6 and the "proper interpretation" of T will be discussed in Section 5.

Another possible transformation is given by

$$z = \frac{R^2 - 1}{4T}, \qquad y(z) = y(R, T). \tag{10}$$

Combining (6), (7), and (10) gives

$$\begin{aligned} \left[\left(z + \frac{1}{4T}\right) D(y)\,y'\right]' + zy' &= 0, \\ y(\infty) = 1, &\qquad T > 0, \\ y(0) = 0, &\qquad T > 0. \end{aligned} \tag{11}$$

In this case the parameter T occurs in the differential equation rather than in the boundary conditions.

From numerical computations we found that the solutions for (9) and (11) were very nearly the same for all values of x, for several different diffusivities, and for all $T \geqslant 0.001$. Thus in the remainder of this paper we will consider only the transformation (8) and the resulting system (9).

4. Linear Problem

In (6) and (9) replace $D(y)$ by

$$D(y) = D_1 = \mathrm{Max}[1, A/2]. \tag{12}$$

The solution of (6), (7), and (12) is given in Carslaw and Jaeger [3] and in Crank [4]. In terms of the notation in this paper the solution is

$$y(R, T) = \frac{2}{\pi} \int_0^\infty A_0(T, u)\, C_0(u, Ru)\, du \tag{13}$$

where

$$A_0(T, u) = \frac{\exp[-D_1 T u^2]}{u[J_0^2(u) + Y_0^2(u)]}, \qquad C_0(u, Ru) = J_0(u)\, Y_0(Ru) - Y_0(u)\, J_0(Ru), \tag{14}$$

and where $J_0(u)$ and $Y_0(u)$ are Bessel functions of order zero and of the first and second kind, respectively.

The amount of diffusing substance crossing a unit area of the surface at $R = 1$ in unit time is proportional to the partial derivative of y with respect to R, evaluated at $R = 1$. This expression is given by

$$I(T) = \frac{4}{\pi^2} \int_0^\infty A_0(T, u)\, du. \tag{15}$$

In the next section we will use $I(T)$ in one of the formulations for the relationship $T = f(T_1)$. In [5], Jaeger and Clark discuss some of the asymptotic properties of $(\pi^2/4)\, I(T)$ and also list a short table of numerical values for it.

Combining (9) and (12) gives

$$D_1(xy')' + (xy') = 0. \tag{16}$$

The solution of (16) which satisfies the boundary conditions in (9) is given by

$$y(x) = 1 - \frac{E_1(x/D_1)}{E_1(1/4D_1T)} \tag{17}$$

where

$$E_1(x) = \int_x^\infty \frac{e^{-\eta}}{\eta}\, d\eta. \tag{18}$$

Tables of values for this well-known exponential integral, $E_1(x)$, are given in Pagurova [6].

5. Parameter T in (9)

The object of this paper is to find a reasonably good approximation to the solution of (6) and (7) for a fixed time T_1 by using the system defined in (9). If this is possible, a great saving in computer time is then feasible because no "marching process" is required to arrive at time T_1, as is the case in solving the partial differential equation. The manner in which the solution of (9) is related to the solution of (6) and (7) for a fixed time T_1 is outlined in the following: replace T in the boundary condition in (9) by $f(T_1)$; solve (9) for y as a function of x; then plot these values of y vs. $R = \sqrt{4f(T_1)x}$; this is the approximation to the solution of (6) and (7) for fixed T_1.

Since T is being considered as a system parameter in (9), a question then arises, "how is T related to T_1, or what is the f in $T = f(T_1)$?". The authors know of no unique f and in fact there probably is none. But through the vehicle of numerical experiment, we found two good candidates.

Probably the most obvious choice of f is the identity function, namely, $f(T_1) = T_1$. This is the choice used in [2] where Drake *et al.* found that the solution of the linear problem for (9) approached the solution of the linear problem for (6) and (7) uniformly in R as $T \to \infty$. But for small T the difference between the two solutions was significant. This was also true for three different nonconstant diffusivities. Thus, a better candidate for f was desired; one which would give a more uniform approximation in T, as well as in R.

The two successful candidates which we consider in this paper are based on the linear problems discussed in Section 4. The first of these candidates for f is obtained by equating the slopes of $y(R, T)$ and $y(x)$ at $R = 1$, for a fixed T_1. Since the functions in (13) and (17) already satisfy common boundary conditions for any T, the above requirement is an added restriction which should result in a better approximation than that obtained for $f(T_1) = T_1$. As we will see in Section 6, this is in fact the case. An implicit formula for this new $f(T_1)$ is obtained from (15) and (17), namely,

$$I(T_1) = \frac{2}{E_1\left[\frac{1}{4D_1 f(T_1)}\right] \exp\left[\frac{1}{4D_1 f(T_1)}\right]}. \tag{19}$$

Given the value of T_1, (19) can be solved iteratively for $f(T_1)$. These values of $f(T_1)$ will then be used for the cases when $D(y)$ is nonconstant.

The second of these candidates for f is determined from the equations

$$y(R_1, T_1) = \frac{9}{10} = y\left(\frac{R_1^2}{4f(T_1)}\right). \tag{20}$$

The reason for using 9/10 (although somewhat arbitrary) is based on the fact that the difference between $y(R, T_1)$ in (13) and $y(R^2/4T_1)$ in (17) is maximum in the neighborhood of $y(R_1, T_1) = 9/10$. Thus Eq. (20) is an added restriction which will force $y(R, T)$ in (13) and $y(R^2/4f(T)$ in (17) to be closer to one another for all R and T. Given T_1, the value of R_1 can be obtained from (13) and (20), namely,

$$\frac{9\pi}{20} = \int_0^\infty A_0(T_1, u)\, C_0(u, R_1 u)\, du. \tag{21}$$

Once R_1 is known, $f(T_1)$ can be obtained from (17) and (20), giving,

$$f(T_1) = \frac{R_1^2}{4D_1 E_1^{-1}\left[\frac{1}{10} E_1\left(\frac{1}{4D_1 f(T_1)}\right)\right]}, \tag{22}$$

where $E_1^{-1}(x)$ is the inverse of $E_1(x)$. In solving for R_1 in (21), the numerical tables given in Jaeger's paper [7] and in Goldenburg's paper [8] are very useful. Equation (22) can be solved for $f(T_1)$ by an iterative procedure.

6. Numerical Solutions

A. *Finite Range for R*

For obvious reasons the upper limit for R in (7) and (9) must be replaced by a finite value of R, say R_2. Thus, the conditions in (7) are redefined as

$$\begin{aligned} y(R, 0) &= 1, \quad 1 < R \leqslant R_2, \\ y(R_2, T) &= 1, \quad T \geqslant 0, \\ y(1, T) &= 0, \quad T \geqslant 0. \end{aligned} \tag{7a}$$

The new system replacing (9) is

$$[xD(y)\,y']' + xy' = 0,$$
$$y\left(\frac{R_2^2}{4T}\right) = 1, \qquad T > 0, \tag{9a}$$
$$y\left(\frac{1}{4T}\right) = 0, \qquad T > 0.$$

The new initial-boundary value problem defined by (6) and (7a) possesses a nontrivial steady-state solution. It is easy to see that the implicit form of this solution is given by

$$R = R_2^{P(y)}, \tag{23}$$

where

$$P(y) = \frac{\tilde{D}(y) - \tilde{D}(0)}{\tilde{D}(1) - \tilde{D}(0)},$$
$$\frac{d\tilde{D}(y)}{dy} = D(y). \tag{24}$$

For a given $D(y)$, the numerical solution of (6) and (7a) should approach the expression in (23) as T becomes large. Thus (23) can be used to partially check the accuracy and stability of a finite-difference scheme used to solve (6) and (7a). It can also be used to check the reliability of using (9a) in place of (6) and (7a).

B. *Difference Schemes for* (6)

We used two implicit finite-difference approximations to (6). The first of these is given by

$$\frac{R_i(y_{i,j+1} - y_{i,j})}{\delta_{j+1}} = \frac{2}{\Delta_{i+1} + \Delta_i}\left[\frac{R_{i+1} + R_i}{2} \cdot D\left(\frac{y_{i+1,j} + y_{i,j}}{2}\right)\frac{y_{i+1,j+1} - y_{i,j+1}}{\Delta_{i+1}} - \frac{R_i + R_{i-1}}{2} D\left(\frac{y_{i,j} + y_{i-1,j}}{2}\right)\frac{y_{i,j+1} - y_{i-1,j+1}}{\Delta_i}\right], \tag{25}$$

where

$$y_{i,j} = y(R_i\,, T_j),$$
$$\Delta_i = R_i - R_{i-1}\,, \tag{26}$$
$$\delta_j = T_j - T_{j-1}\,.$$

If $\Delta_{i+1} = \Delta_i = \Delta R$, $\delta_{j+1} = \Delta T$, R_{i+1} and R_{i-1} are replaced by R_1, and $D(y) = \sigma =$ constant, then (25) reduces to formula three listed on p. 189 of [9]. For this reduced system the difference equation is always stable and the truncation error is defined by

$$e = O(\Delta T) + O[(\Delta R)^2]. \tag{27}$$

The difference equation in (25) was solved by an iterative point method; see [10, pp. 24, 25]. The first approximation is obtained from the matrix equation resulting from (25) and all latter approximations are obtained from a Gauss–Seidel iteration scheme, given below:

$$y_{i,j+1}^{(n+1)} = A_{ij} y_{i-1,j+1}^{(n+1)} + B_{ij} y_{i+1,j+1}^{(n)} + C_{ij}, \tag{28}$$

where

$$\left.\begin{aligned}
A_{ij} &= Q_{ij}P_{ij}/L_{ij},\\
B_{ij} &= Q_{ij}P_{i+1,j}/L_{ij},\\
C_{ij} &= y_{i,j}^{(1)}/L_{ij},\\
L_{ij} &= 1 + Q_{ij}P_{ij} + Q_{ij}P_{i+1,j},\\
Q_{ij} &= \delta_{j+1}/R_i(\Delta_{i+1} + \Delta_i),\\
P_{ij} &= \frac{R_i + R_{i-1}}{\Delta_i} D\left(\frac{y_{ij}^{(1)} + y_{i-1,j}^{(1)}}{2}\right),\\
y_{i,j}^{(n)} &= n\text{th iterate of } y(R_i, T_j).
\end{aligned}\right\} \tag{29}$$

This iteration scheme, (28) and (29), uses the most recent iterates as soon as they become available; this improves the rate of convergence of the process as opposed to methods which do not do this.

The other finite-difference scheme which we used was motivated by formula nine on p. 190 of [9] and by the treatment of the nonlinear equation

$$u_t = (u^5)_{xx} \tag{30}$$

on pp. 201–203 of the same reference. The resulting difference equation is given by

$$\begin{aligned}
&\frac{3R_i}{2\delta_{j+1}}(y_{i,j+1} - y_{i,j}) - \frac{R_i}{2\delta_j}(y_{i,j} - y_{i,j-1})\\
&\quad = \frac{2}{\Delta_{i+1} + \Delta_i}\left[\frac{R_{i+1/2}}{\Delta_{i+1}}\{[\tilde{D}]_{i+1,j} + [D]_{i+1,j}(y_{i+1,j+1} - y_{i+1,j}) - [\tilde{D}]_{ij}\right.\\
&\qquad - [D]_{ij}(y_{i,j+1} - y_{i,j})\} - \frac{R_{i-1/2}}{\Delta_i}\{[\tilde{D}]_{ij} + [D]_{ij}(y_{i,j+1} - y_{ij})\\
&\qquad \left. - [\tilde{D}]_{i-1,j} - [D]_{i-1,j}(y_{i-1,j+1} - y_{i-1,j})\}\right]
\end{aligned} \tag{31}$$

where

$$[D]_{ij} = D(y_{ij}),$$
$$\frac{d\tilde{D}}{dy} = D(y), \tag{32}$$
$$[\tilde{D}]_{ij} = \tilde{D}(y_{ij}).$$

If $\Delta_{i+1} = \Delta_i = \Delta R$, $\delta_{j+1} = \delta_j = \Delta T$, $R_{i+1/2}$ and $R_{i-1/2}$ are replaced by R_i, $D(y) = \sigma =$ constant, and $\tilde{D}(y) = \sigma y$, then (31) reduces to formula nine on p. 190 in [9]. For this reduced system the difference equation is always stable and the truncation error is defined by

$$e = O[(\Delta T)^2] + O[(\Delta R)^2]. \tag{33}$$

The reason that the expression $\tilde{D}(y)$ is present in (31) is that (6) was written as

$$R y_T = [R\{\tilde{D}(y)\}_R]_R \tag{34}$$

and then differenced in the same manner as (30), see [9]. Thus, considering (33) and the form of (34), one would expect that (31) is a higher order difference equation in time than (25), and might expect (31) to give some improvement in the space differencing.

A Gauss–Seidel iteration, similar to that given in (28), was used to solve the difference equation in (31).

C. *Difference Scheme for* (9a)

Since (9a) defines a two-point boundary value problem, a "shooting method" approach was used in its numerical solution. The difference equation is

$$x_j(y_{j+1}^{(n+1)} - y_{j-1}^{(n+1)}) + \frac{x_{j+1} + x_j}{x_{j+1} - x_j}(y_{j+1}^{(n+1)} - y_j^{(n+1)})\, D\left(\frac{y_{j+1}^{(n)} + y_j^{(n)}}{2}\right)$$
$$- \frac{x_j + x_{j-1}}{x_j - x_{j-1}}(y_j^{(n+1)} - y_{j-1}^{(n+1)})\, D\left(\frac{y_j^{(n)} + y_{j-1}^{(n)}}{2}\right) = 0 \tag{35}$$

where $y_j^{(n)}$ is the nth iterate of $y(x_j)$. The initial guess was taken to be

$$y_j^{(1)} = \frac{\ln R_j}{\ln R_2}. \tag{36}$$

D. *Specific Examples*

In order to test the central idea of this paper, namely, replacing (6) and (7a) by (9a), we considered twenty-two different expressions for $D(y)$. The $D(y)$'s which we considered are as follows: (1) $D(y) = 1 + (A - 1)y^P$ and $D(y) = A - (A - 1)y^P$

for $P = 1/4$, $1/3$, $1/2$, 1, 2, 3, and 4; (2) $D(y) = \max[1, A/2]$; (3) $D(y) = A^y$ and A^{1-y}; (4) three diffusivities containing an inflection point; and (5) two logarithmic diffusivities. For the numerical computations the value of A was taken to be 200, which is a representative value for soil moisture problems (see Table I in [2]); and the value for R_2 was taken to be 1000.

Even though the linear analogues of the differencing systems used in this study are always stable, this is not always the case for nonlinear systems. Thus, to insure stability and to obtain good resolution, the following variable time and space increments were used: $0 \leqslant T \leqslant 0.004$, $\Delta T = 0.0001$; $0.004 \leqslant T \leqslant 1$, $\Delta T = 0.001$; $1 \leqslant T \leqslant 100$, $\Delta T = 0.01$; $100 \leqslant T \leqslant 500$, $\Delta T = 0.1$; $500 \leqslant T \leqslant 20{,}000$, $\Delta T = 10$; $1 \leqslant R \leqslant 10$, $\Delta R = 0.1$; $10 \leqslant R \leqslant 100$, $\Delta R = 1$; $100 \leqslant R \leqslant 1000$, $\Delta R = 10$.

Since our main objective was to compare the solutions of (9a) with those of (6) and (7a), we were conservative in our choice of time and space increments and in the number of iterations used in the solutions of the difference equations. In (25) and (31), we used 3 iterations (when 1 or 2 were sufficient); and in (35) and (36), we used 20 iterations (when 10 would have been enough).

E. *Solutions for $f(T_1)$*

Figure 1 gives the three relationships between T_1 and $f(T_1)$ discussed in Section 5. In solving (19) for $f(T_1)$, the tables for $I(T_1)$ given in [5] and the tables for $E_1(X)e^X$ given in [6] were used. The tables and asymptotic values presented in [7] and [8] were used to solve for R_1 in Eq. (21). Given the values of R_1, (22) was solved for $f(T_1)$ using the tables for $E_1(X)$ given in [11].

F. *Comparison of Solutions for the Original and Transformed Systems*

In our study, as mentioned above, we considered twenty-two different expressions for the diffusivity, $D(y)$. Space limitations allow us to only report some representative values; these are given in Tables I, II, and III. The results for the diffusivities not reported in this paper are similar to those discussed below with no exception.

In Table I, the numerical solutions of (6) and (7a) are compared with those of (9a) for various values of R and T and for three different diffusivities. The diffusivity $D = 200^y$ is one of the extreme "sink" cases considered in this study; $D = 200^{1-y}$ is one of the extreme "source" cases; and $D = 1 + 199y^{1/4}$ gives results which are nearly the same as those for $D = 100$. The numerical values denoted by P.D.E. are from the solutions of (6) and (7a) based on (25) and (28). For the transformed system, (9a), three sets of values are given. The values denoted by $f = T_1$, f_{19}, and f_{22} are, respectively, those given by the numerical solution of (9a) for $f(T_1) = T_1$, $f(T_1)$ given by (19), and $f(T_1)$ given by (22).

When $f(T_1) = T_1$, the solutions of the transformed system approach those

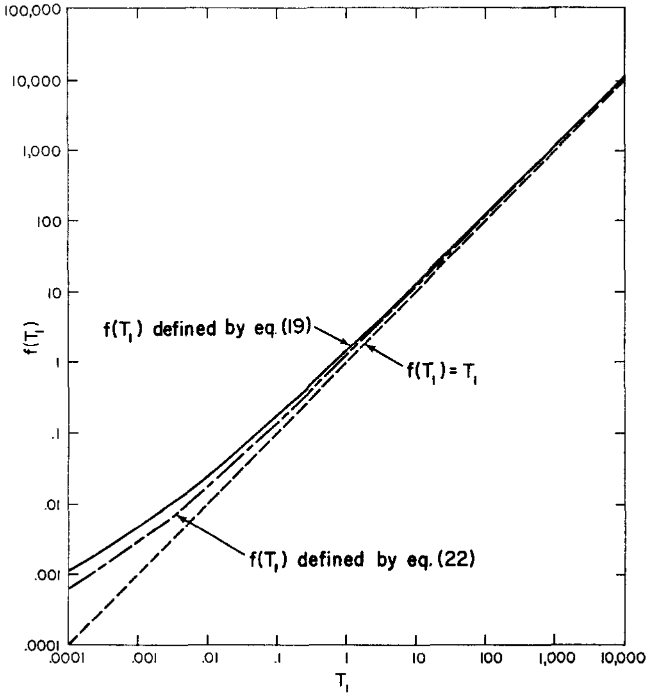

Fig. 1. Relationships for $f(T_1)$ vs. T_1.

TABLE I

Comparison of Solutions for Various T's and Various Diffusivities

For $D = 200^y$ and $T = 0.01$				
	Values of $y(R, T)$			
R	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.709	0.818	0.729	0.761
1.4	0.800	0.876	0.805	0.831
1.6	0.851	0.910	0.850	0.872
1.8	0.885	0.933	0.880	0.900
2	0.909	0.949	0.902	0.920
4	0.990	0.996	0.981	0.988
For $D = 200^{1-y}$ and $T = 0.01$				
1.2	0.067	0.122	0.067	0.082
1.4	0.150	0.336	0.147	0.190
1.6	0.271	1.000	0.253	0.361
1.8	0.589	1.000	0.421	0.814
2	0.990	1.000	0.813	1.000
4	1.000	1.000	1.000	1.000

TABLE 1 (continued)

For $D = 1 + 199y^{1/4}$ and $T = 0.01$

R	Values of $y(R, T)$			
	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.215	0.274	0.207	0.229
1.4	0.353	0.439	0.338	0.371
1.6	0.460	0.559	0.438	0.479
1.8	0.548	0.651	0.518	0.564
2	0.621	0.723	0.584	0.634
4	0.945	0.977	0.900	0.936

For $D = 200^y$ and $T = 10$

R	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.439	0.444	0.439	0.440
1.4	0.540	0.545	0.540	0.541
1.6	0.598	0.603	0.597	0.599
1.8	0.637	0.642	0.637	0.638
2	0.667	0.672	0.666	0.667
4	0.793	0.798	0.792	0.794
8	0.867	0.872	0.867	0.868
20	0.934	0.938	0.933	0.935
40	0.970	0.973	0.969	0.970
80	0.992	0.994	0.991	0.992

For $D = 200^{1-y}$ and $T = 10$

R	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.012	0.012	0.012	0.012
1.4	0.022	0.023	0.022	0.022
1.6	0.032	0.033	0.032	0.032
1.8	0.040	0.042	0.041	0.040
2	0.049	0.051	0.049	0.049
4	0.115	0.120	0.114	0.116
8	0.217	0.230	0.215	0.217
20	0.617	0.721	0.592	0.618
40	1.000	1.000	1.000	1.000
80	1.000	1.000	1.000	1.000

For $D = 1 + 199y^{1/4}$ and $T = 10$

R	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.078	0.079	0.078	0.078
1.4	0.128	0.131	0.129	0.128
1.6	0.168	0.172	0.169	0.168
1.8	0.202	0.206	0.203	0.201
2	0.230	0.235	0.230	0.231
4	0.403	0.412	0.402	0.404
8	0.558	0.569	0.557	0.560
20	0.745	0.759	0.743	0.747
40	0.872	0.883	0.867	0.871
80	0.966	0.971	0.961	0.963

TABLE 1 (continued)

For $D = 200^{y}$ and $T = 10{,}000$

R	Values of $y(R, T)$			
	P.D.E.	$f = T_1$	f_{19}	f_{22}
1.2	0.353	0.354	0.354	0.354
1.4	0.451	0.451	0.451	0.451
1.6	0.508	0.508	0.508	0.508
1.8	0.547	0.546	0.546	0.546
2	0.576	0.575	0.575	0.575
4	0.702	0.701	0.700	0.700
8	0.776	0.776	0.776	0.775
20	0.844	0.844	0.844	0.844
40	0.883	0.883	0.883	0.883
80	0.915	0.916	0.916	0.916
200	0.950	0.952	0.951	0.951
400	0.973	0.975	0.974	0.974
800	0.994	0.994	0.994	0.994
For $D = 200^{1-y}$ and $T = 10{,}000$				
1.2	0.006	0.006	0.006	0.006
1.4	0.011	0.011	0.011	0.011
1.6	0.015	0.015	0.015	0.015
1.8	0.019	0.019	0.019	0.019
2	0.022	0.023	0.022	0.022
4	0.048	0.048	0.048	0.048
8	0.077	0.078	0.077	0.077
20	0.125	0.126	0.124	0.124
40	0.171	0.173	0.170	0.170
80	0.232	0.236	0.231	0.231
200	0.363	0.372	0.362	0.361
400	0.601	0.627	0.593	0.591
800	1.000	1.000	1.000	1.000
For $D = 1 + 199y^{1/4}$ and $T = 10{,}000$				
1.2	0.053	0.054	0.053	0.053
1.4	0.088	0.088	0.088	0.088
1.6	0.115	0.116	0.116	0.116
1.8	0.138	0.139	0.139	0.139
2	0.158	0.159	0.159	0.159
4	0.276	0.277	0.277	0.277
8	0.382	0.384	0.384	0.384
20	0.512	0.515	0.515	0.515
40	0.605	0.609	0.609	0.609
80	0.694	0.699	0.699	0.699
200	0.808	0.814	0.813	0.813
400	0.892	0.897	0.897	0.897
800	0.974	0.976	0.976	0.976

given by (6) and (7a) as T becomes large. For the other two choices of $f(T_1)$, the transformed solutions and those of (6) and (7a) agree more uniformly with respect to T, as well as R. The choice of f_{19} as opposed to f_{22} depends on whether one wants the best agreement between the original and transformed systems in the neighborhood of $R = 1$, or in the neighborhood of $y = 9/10$.

In Table II, numerical values for the steady solutions of (6) and (7a) are given for the same three diffusivities. These values were obtained from (23) and (24). Comparing the values for $T = 10{,}000$ in Table I with those in Table II we see that for $D = 200^y$ and $D = 1 + 199y^{1/4}$ the solutions at $T = 10{,}000$ are for all practical purposes the "steady-state" solutions. As was the case for some of the other "source" problems, the solution for $D = 200^{1-y}$ takes much longer to approach the steady-state solution. But if the computations are continued long enough, the steady-state solutions are approached for all the diffusivities studied.

TABLE II

Steady-State Values for Various Diffusivities

	Values of $y(R, T)$		
R	$D = 200^y$	$D = 200^{1-y}$	$D = 1 + 199y^{1/4}$
1.2	0.348	0.006	0.054
1.4	0.448	0.009	0.089
1.6	0.506	0.013	0.116
1.8	0.546	0.017	0.139
2	0.575	0.020	0.159
4	0.701	0.042	0.276
8	0.776	0.067	0.382
20	0.843	0.107	0.512
40	0.883	0.143	0.605
80	0.915	0.189	0.694
200	0.950	0.272	0.808
400	0.973	0.375	0.892
800	0.994	0.622	0.974

In Table III, the two difference schemes for (6) and (7a) given in (25) and (31) are compared for $D = 1 + 199y^4$ and $D = 200^{1-y}$. Since scheme (31) is slower than (25) and since the two schemes give nearly the same results for $D = 200^{1-y}$, scheme (25) was used for most of the above computations. For $D = 1 + 199y^4$, the system given by (31) appears to be more accurate since its numerical solution checks the steady-state solution closer than that given by (25). From the above discussion concerning (27), (33), and (34), one might expect this result.

TABLE III

Comparison of Difference Formulas (25) and (31)

R	T = 0.01 (25)	T = 0.01 (31)	T = 1 (25)	T = 1 (31)	T = 100 (25)	T = 100 (31)	T = 10,000 (25)	T = 10,000 (31)	Steady state
				Values of $y(R, T)$ for $D = 1 + 199y^4$					
1.2	0.765	0.687	0.602	0.541	0.513	0.470	0.471	0.438	0.439
1.4	0.825	0.781	0.661	0.624	0.575	0.550	0.536	0.517	0.518
1.6	0.864	0.834	0.700	0.672	0.614	0.595	0.575	0.561	0.561
1.8	0.892	0.871	0.728	0.706	0.641	0.626	0.603	0.591	0.591
2	0.914	0.897	0.750	0.731	0.662	0.649	0.624	0.614	0.615
4	0.990	0.989	0.854	0.846	0.761	0.755	0.721	0.716	0.716
8	1.000	1.000	0.921	0.918	0.826	0.822	0.783	0.780	0.780
20	1.000	1.000	0.980	0.979	0.889	0.886	0.844	0.842	0.842
40	1.000	1.000	0.998	0.998	0.926	0.925	0.880	0.879	0.880
80	1.000	1.000	1.000	1.000	0.958	0.957	0.912	0.911	0.911
200	1.000	1.000	1.000	1.000	0.989	0.989	0.948	0.947	0.947
400	1.000	1.000	1.000	1.000	0.999	0.999	0.972	0.971	0.971
800	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.993	0.993
				Values of $y(R, T)$ for $D = 200^{1-y}$					
1.2	0.067	0.068	0.017	0.017	0.009	0.009	0.006	0.006	0.006
1.4	0.150	0.152	0.033	0.033	0.016	0.016	0.011	0.011	0.009
1.6	0.271	0.271	0.048	0.048	0.023	0.023	0.015	0.015	0.013
1.8	0.589	0.484	0.062	0.063	0.029	0.029	0.019	0.019	0.017
2	0.990	0.961	0.076	0.076	0.035	0.035	0.022	0.022	0.020
4	1.000	1.000	0.205	0.206	0.079	0.079	0.048	0.048	0.042
8	1.000	1.000	0.626	0.626	0.135	0.135	0.077	0.077	0.067
20	1.000	1.000	1.000	1.000	0.252	0.252	0.125	0.125	0.107
40	1.000	1.000	1.000	1.000	0.439	0.439	0.171	0.171	0.143
80	1.000	1.000	1.000	1.000	0.999	0.999	0.232	0.232	0.189
200	1.000	1.000	1.000	1.000	1.000	1.000	0.363	0.364	0.272
400	1.000	1.000	1.000	1.000	1.000	1.000	0.601	0.601	0.375
800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.622

For the time and space increments given above and for all the diffusivities studied, the difference schemes for the original system and the transformed system were numerically stable. Also, the difference systems were all asymptotically stable, in the numerical sense, with respect to the steady-state solution. That is, once the numerical solution became stationary in time it would not move from this stationary solution, no matter how many time steps were taken beyond this point. This was true for the difference schemes for (6) and (7a) and for (9a). The stationary solutions were in all cases very nearly equal to the steady-state solutions given by (23) and (24).

7. Conclusions

The main result of this paper is that the initial-boundary value problem given by (6) and (7a) can be replaced, in an approximate sense, by the two-point boundary value problem in (9a) through the vehicle of the similarity transformation defined by (8). Since time T occurs in a boundary condition in (9a), there is no similarity solution of (6) and (7a) of the type specified in (8). But if T in (9a) is interpreted as a parameter in a "proper way," the numerical solutions of (9a) are in very good agreement with those of (6) and (7a) for a large class of diffusivities, $D(y)$. This method of approximating the numerical solution of (6) and (7a) for a given time T_1 saves a great deal of computer time since a "marching process" is not required to get to time T_1, as is the case in solving the partial differential equation.

In validating the above numerical technique, two difference schemes for solving the initial-boundary value problem in (6) and (7a) were proposed, see (25) and (31). These difference schemes were stable for the diffusivities considered in this study. Also, since a finite range was required for the space variable in the numerical calculations, the system defined by (6) and (7a) possesses a nontrivial steady-state solution. The two difference schemes are asymptotically stable in a numerical sense with respect to these steady-state solutions. This is also true of the numerical solutions for (9a).

Another important application of the present study is that certain qualitative results about the solutions of (9) can be obtained rather easily from the differential equation itself. These results, in turn, can be applied to the solutions of (6) and (7), at least in an approximate sense. For example in [2], Drake *et al.* proved some theorems concerning various monotonicity properties of the transformed system. These properties carry over to the original system for certain forms of the diffusivity.

The type of "local similarity" considered in this paper is related to the local similarity of boundary layer theory (see [12]) where the time variable in [12] is replaced by the distance along the body. The authors of this paper feel that the technique considered here can be applied to many other problems in physics. One problem which the authors are considering currently is the vertical drainage problem in ground water flow; and in the future, the coagulation equation of aerosol and cloud physics will be considered.

The technique considered in this paper can be summarized in the following way:

1. A physical problem is formulated in terms of a partial differential equation (or possibly an integrodifferential equation) and certain auxiliary conditions, such as initial and boundary conditions. The equation also is dependent upon certain system parameters. In the radial flow problem the system parameter is the soil moisture diffusivity $D(y)$. In order that the physical problem be mathematically well-posed, the solutions of the system should be continuously dependent upon

the initial and boundary conditions and the system parameters (see [13] and [14]).

2. The original system is assumed to satisfy some but not all of the requirements for the existence of a similarity solution. The transformation corresponding to this similarity solution is applied to the original system. Any "stray" independent variable in the new system is converted to a time parameter, $f(T_1)$, through the use of the similarity transformation.

3. The nonlinearity of the original and transformed system is due to the variation of the system parameters with the dependent variable, for example, $D(y)$ in Eq. (6).

4. The time parameter $f(T_1)$ is determined from the corresponding linear systems, that is, the systems corresponding to constant parameters. The actual choice of $f(T_1)$ depends on the region where one wants the best agreement between the original and transformed systems. For example, in the present problem if one is most interested in the rate of inflow or outflow at $R = 1$, the $f(T_1)$ given by (19) should be used.

5. As in the present problem, this technique will be most economical for problems which must be solved repeatedly for different formulations of the system parameters. In the radial flow of soil moisture, diffusivities are experimentally determined and their formulations are given graphically or by empirically determined formulas. Thus, the problem discussed in this paper will be solved many more times in the future. Therefore, there is a question concerning the accuracy of the present technique of solving this problem.

6. For the radial flow problem, and for other well-posed physical problems, the continuous dependence of the solutions on the system parameters can be used to advantage in checking the accuracy of the local similarity concept for the various formulations of the parameters anticipated in a given problem. The authors of this paper feel that the numerical checks given for the three widely different diffusivities in Table I are sufficient for assessing the quantitative errors which will occur in applying the local similarity concept to the radial flow problem for the various formulations of $D(y)$ which a soil physicist may consider.

7. Our recommendation for the application of this technique to other physical problems is to check the accuracy of the approximation for the "extremes" of the system parameters and for one slightly nonlinear case. If the accuracy of these results is allowable for the physical problem under consideration, the "continuous dependence property" of the system will allow one to use the local similarity concept for other formulations of the system parameters. In the radial flow problem, $D = 200^y$ is an extreme "sink" case; $D = 200^{1-y}$ is an extreme "source" case; and $D = 1 + 199y^{1/4}$ is a "slightly nonlinear" case which gives results nearly equal to those for $D = 100$.

Finally, the authors of this paper wish to report that we did check the local similarity concept for a family of diffusivities, twenty in number, which vary in a "continuous manner" from $D = 200^y$ to $D = 200^{1-y}$. The results of this study verify the statements made in items 6 and 7 of the above summary. (These numerical solutions will be published elsewhere.) Thus, we stand by our recommendation in item 7 whenever this technique is applied to other problems.

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